

Separability preserving Dirac reductions of Poisson pencils on Riemannian manifolds

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Abstract

Dirac deformation of Poisson operators of arbitrary rank is considered. The question when Dirac reduction does not destroy linear Poisson pencils is studied. A class of separability preserving Dirac reductions in the corresponding quasi-bi-Hamiltonian systems of Benenti type is discussed. Two examples of such reductions are given. This paper will appear in J. Phys. A: Math. Gen.

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1 Introduction

Recently a new (quasi)-bi-Hamiltonian separability theory of Liouville integrable finite dimensional systems was constructed [1]-[15] based on general properties of Poisson pencils on manifolds. A natural further step within this theory is to investigate admissible integrable and separable reductions of these integrable / separable systems onto appropriate submanifolds. The most natural approach seems to be the one based on the Dirac theory of constrained dynamics [16],[17]. The presented paper contains only some special cases of such reductions, but even in these cases the problem is far from being trivial. The difficulties we met during our research inclined us to reconsider the Dirac formalism from the point

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of view of Poisson bivectors rather than from the point of view of constrained dynamics.

The paper is organized as follows. In this introductory part we remind basic concepts of Poisson geometry and of (quasi-)bi-Hamiltonian systems. In Section 2 we formulate the theory of Dirac reductions of Poisson brackets in terms of Poisson bivectors. Our construction is more general than usually met in literature since we consider reduction procedure on the whole *foliation* of submanifolds. Here we also explain that our approach indicates that Dirac classification of constraints onto those of "first" and of "second class" requires further discussion as our further results show. In Section 3 we review the recent results on separability theory of quasi-bi-Hamiltonian systems of Benenti type. In section 4 we perform Dirac reduction of Poisson pencil and corresponding quasi-bi-Hamiltonian chain of Benenti type onto a particularly chosen submanifold. Chosen constraints preserve the Liouville integrability as well as the coordinates of separation of considered system. The main obstacle of such a choice is that these constraints are nonexpressible in natural (original) coordinates. Hence, in Section 5 we modify constraints introduced in the previous section and obtain an equivalent reduction, that is expressible directly in original coordinates. Finally, in Section 6 we illustrate the results by two nontrivial examples of constrained separable dynamics.

Let us first remind few basic facts from Poisson geometry. Given a manifold \mathcal{M} , a *Poisson operator* π on \mathcal{M} is a mapping $\pi : T^*\mathcal{M} \rightarrow T\mathcal{M}$ that is fibre-preserving (i.e. $\pi|_{T_x^*\mathcal{M}} : T_x^*\mathcal{M} \rightarrow T_x\mathcal{M}$ for any $x \in \mathcal{M}$) and such that the induced bracket on the space $C^\infty(\mathcal{M})$ of all smooth real-valued functions on \mathcal{M}

$$\{.,.\}_\pi : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M}) \quad , \quad \{F, G\}_\pi \stackrel{\text{def}}{=} \langle dF, \pi dG \rangle \quad (1)$$

(where $\langle ., . \rangle$ is the dual map between $T\mathcal{M}$ and $T^*\mathcal{M}$) is skew-symmetric and satisfies Jacobi identity (the bracket (1) always satisfies the Leibniz rule $\{F, GH\}_\pi = G\{F, H\}_\pi + H\{F, G\}_\pi$). Throughout the whole paper the symbol d will denote the operator of exterior derivative. The operator π can always be interpreted as a bivector, $\pi \in \Lambda^2(\mathcal{M})$ and in a given coordinate system (x^1, \dots, x^m) on \mathcal{M} we have

$$\pi = \sum_{i < j}^m \pi^{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

A function $C : \mathcal{M} \rightarrow \mathbb{R}$ is called *Casimir function* of the Poisson operator π if for arbitrary function $F : \mathcal{M} \rightarrow \mathbb{R}$ we have $\{F, C\}_\pi = 0$ (or, equivalently, if $\pi dC = 0$). A linear combination $\pi_\xi = \pi_1 - \xi\pi_0$ ($\xi \in \mathbb{R}$) of two Poisson operators π_0 and π_1 is called *Poisson pencil* if the operator π_ξ is Poisson for any value of the parameter ξ . In this case we say that π_0 and π_1 are *compatible*. Given a Poisson pencil $\pi_\xi = \pi_1 - \xi\pi_0$ we can often construct a sequence of vector fields Y_i on \mathcal{M} that have a twofold Hamiltonian form (a so called *bi-Hamiltonian chain*)

$$Y_i = \pi_1 dh_i = \pi_0 dh_{i+1} \quad (2)$$

where $h_i : \mathcal{M} \rightarrow \mathbb{R}$ are called Hamiltonians of the chain (2) and where i is

some discrete index. This sequence of vector fields may or may not truncate (depending on existence of Casimir functions). In the case when the Poisson pencil π_ξ is degenerated but projectable into a symplectic leaf \mathcal{N} (of dimension $2n$) of π_0 the bi-Hamiltonian chain (2) on \mathcal{M} turns into a so called quasi-bi-Hamiltonian chain on \mathcal{N} of the form

$$\theta_1 dH_i = \theta_0 dH_{i+1} + \sum_{\substack{j=1 \\ j \neq i+1}}^n \alpha_{ij} \theta_0 dH_j, \quad i = 1, \dots, n, \quad H_{n+1} \equiv 0, \quad (3)$$

where θ_i are projections of π_i onto \mathcal{N} , the functions H_j are restrictions of h_j to \mathcal{N} : $H_j = h_j|_{\mathcal{N}}$, and α_{ij} are some multipliers (real functions). And vice versa: having a quasi-bi-Hamiltonian chain (3) on the manifold \mathcal{N} one can lift it to a bi-Hamiltonian chain (2) on the extended manifold \mathcal{M} . (Quasi-)bi-Hamiltonian chains (called also (quasi-)bi-Hamiltonian systems) possess very interesting differential-algebraic properties and are one of key notions in the theory of integrable systems, due to the fact that in many cases the systems (2) and (3) are Liouville integrable [18]. Recently, much effort has been spent in order to exploit the procedure of solving these systems by the method of separation of variables [2]-[15]. In this article we will mainly work with the quasi-bi-Hamiltonian chains (3) rather than bi-Hamiltonian ones, since the pencil $\theta_\xi = \theta_1 - \xi \theta_0$ is always non-degenerated.

2 Dirac reduction of Poisson bivectors

We begin by considering the Dirac reduction procedure in a more general setting that is usually met in literature. Let π be a Poisson bivector, in general degenerated on some manifold \mathcal{M} . Let $\mathcal{S} = \mathcal{F}_0$ be a submanifold in a foliation \mathcal{F} of the manifold \mathcal{M} defined by m functionally independent functions (constraints) $\varphi_i : \mathcal{M} \rightarrow \mathbb{R}$, $i = 1, \dots, m$.

$$\mathcal{F}_s = \{x \in \mathcal{M} : \varphi_i(x) = s_i, i = 1, \dots, m\}$$

Thus, \mathcal{S} is a submanifold of codimension m in \mathcal{M} . Moreover, let Z_i , $i = 1, \dots, m$ be some vector fields transversal to \mathcal{F}_s , spanning a regular distribution \mathcal{Z} in \mathcal{M} of constant dimension m (that is a smooth collection of m -dimensional subspaces $\mathcal{Z}_x \subset T_x \mathcal{M}$ at every point x in \mathcal{M}). The word 'transversal' means here that no vector field Z_i is at any point tangent to the submanifold \mathcal{F}_s passing through this point. Hence, the tangent bundle $T\mathcal{M}$ splits into a direct sum

$$T\mathcal{M} = T\mathcal{F} \oplus \mathcal{Z}$$

(which means that at any point x in \mathcal{M} we have $T_x \mathcal{M} = T_x \mathcal{F}_s \oplus \mathcal{Z}_x$ with s such that $x \in \mathcal{F}_s$) and so does its dual

$$T^* \mathcal{M} = T^* \mathcal{F} \oplus \mathcal{Z}^*,$$

where $T^*\mathcal{F}$ is the annihilator of \mathcal{Z} and \mathcal{Z}^* is the annihilator of $T\mathcal{F}$. That means that if α is a one form in $T^*\mathcal{F}$ then $\alpha(Z_i) = 0$ for all $i = 1, \dots, m$ and if β is a one-form in \mathcal{Z}^* then β vanishes on all vector fields tangent to \mathcal{F}_s . Moreover, we assume that the vector fields Z_i which span \mathcal{Z} are chosen in such a way that $d\varphi_i$, $i = 1, \dots, m$ is a basis in \mathcal{Z}^* that is dual to the basis Z_i of the distribution \mathcal{Z} ,

$$\langle d\varphi_i, Z_j \rangle = Z_j(\varphi_i) = \delta_{ij}, \quad (4)$$

(this is no restriction since for any distribution \mathcal{Z} transversal to \mathcal{F}_s we can choose its basis so that (4) is satisfied). Finally, let us define m vector fields X_i on \mathcal{M} and m^2 functions $\varphi_{ij} : \mathcal{M} \rightarrow \mathbb{R}$ on \mathcal{M} through

$$X_i = \pi(d\varphi_i), \quad \varphi_{ij} = \{\varphi_i, \varphi_j\}_\pi = \langle d\varphi_i, \pi d\varphi_j \rangle = X_j(\varphi_i). \quad (5)$$

The functions φ_{ij} define an m -dimensional skew-symmetric matrix $\varphi = (\varphi_{ij})$, $i, j = 1, \dots, m$. It can be easily shown that

$$[X_j, X_i] = X_{\{\varphi_i, \varphi_j\}_\pi} = \pi d\{\varphi_i, \varphi_j\}_\pi = \pi d\varphi_{ij}, \quad (6)$$

where $\{.,.\}_\pi$ is a Poisson bracket defined by our Poisson bivector π and $[X, Y] = L_X Y = X(Y) - Y(X)$ is the Lie bracket (commutator) of the vector fields X, Y .

A very special choice of our transversal vector fields Z_i originates by taking linear combinations of fields X_j with coefficients *being the entries of the matrix* φ^{-1} .

$$Z_i = \sum_{j=1}^m (\varphi^{-1})_{ji} X_j \quad i = 1, \dots, m. \quad (7)$$

Since the constraint functions φ_i are functionally independent, the vector fields Z_i in (7) will indeed be transversal to the foliation \mathcal{F} . Moreover, they will automatically satisfy the orthogonality condition (4) as $Z_i(\varphi_j) = \sum_{k=1}^m (\varphi^{-1})_{ki} X_k(\varphi_j) = \sum_{k=1}^m (\varphi^{-1})_{ki} \varphi_{jk} = \delta_{ij}$.

Let us now consider the following *deformation* (modification) of the bivector π :

$$\pi_D = \pi - \frac{1}{2} \sum_{i=1}^m X_i \wedge Z_i, \quad (8)$$

where \wedge denotes the wedge product in the algebra of multivectors. This new bi-vector π_D can be properly restricted to \mathcal{F}_s for any $s \in \mathbb{R}^m$ (and thus also to $\mathcal{S} = \mathcal{F}_0$), since the image of π_D considered on a given leaf \mathcal{F}_s of the foliation \mathcal{F} lies in $T\mathcal{F}_s$ for all s . This is the content of the following theorem.

Theorem 1 *Suppose $x \in \mathcal{F}_s$. Then for any $\alpha \in T_x^* \mathcal{M}$ the vector $\pi_D(\alpha)$ is tangent to \mathcal{F}_s i.e. $\pi_D(T_x^* \mathcal{M}) \subset T_x \mathcal{F}_s$.*

Proof. We will show, that $\pi_D(d\varphi_k) = 0$, $k = 1, \dots, m$, since it means that the constraints φ_i , $i = 1, \dots, m$ are Casimirs of π_D (and so are then $\varphi_i - s_i$) which obviously implies the thesis of the theorem. Using the definition (8) of

π_D , the obvious fact that $(X_i \wedge Z_i)d\varphi_k = (X_i \otimes Z_i)d\varphi_k - (Z_i \otimes X_i)d\varphi_k = \langle d\varphi_k, Z_i \rangle X_i - \langle d\varphi_k, X_i \rangle Z_i = Z_i(\varphi_k)X_i - X_i(\varphi_k)Z_i$ we have

$$\begin{aligned}\pi_D(d\varphi_k) &= \pi(d\varphi_k) - \frac{1}{2} \sum_i Z_i(\varphi_k)X_i + \frac{1}{2} \sum_i X_i(\varphi_k)Z_i \\ &= X_k - \frac{1}{2} \sum_i \delta_{ik}X_i + \frac{1}{2} \sum_{i,j} \varphi_{ki}(\varphi^{-1})_{ji}X_j \\ &= X_k - \frac{1}{2}X_k - \frac{1}{2} \sum_i \delta_{ki}X_i = 0\end{aligned}$$

due to the fact that $\varphi_{ij} = -\varphi_{ji}$. ■

Theorem 2 *The bivector π_D in (8) with Z_i as in (7) satisfies the Jacobi identity.*

Proof. It is easy to check that our operator π_D defines the following bracket on \mathcal{M}

$$\{F, G\}_{\pi_D} = \{F, G\}_{\pi} - \sum_{i,j=1}^m \{F, \varphi_i\}_{\pi}(\varphi^{-1})_{ij} \{\varphi_j, G\}_{\pi}, \quad (9)$$

(where $F, G : \mathcal{M} \rightarrow \mathbb{R}$ are two arbitrary functions on \mathcal{M}) which is just the well known *Dirac deformation* [16] of the bracket $\{.,.\}_{\pi}$ associated with π , and as was shown by Dirac [17] it satisfies the Jacobi identity. ■

Remark 3 *If $C : \mathcal{M} \rightarrow \mathbb{R}$ is a Casimir function of π , then it is also a Casimir function of π_D , since in this case (9) yields*

$$\{F, C\}_{\pi_D} = \{F, C\}_{\pi} - \sum_{i,j=1}^m \{F, \varphi_i\}_{\pi}(\varphi^{-1})_{ij} \{\varphi_j, C\}_{\pi} = 0 - 0 = 0.$$

We also know from Theorem 1 that the constraints φ_i are Casimirs of the deformed operator π_D . Thus, we can informally state that *Dirac deformation preserves all the old Casimir functions and introduces new Casimirs φ_i .*

It is now possible to restrict our Poisson operator π_D (or our Poisson bracket $\{.,.\}_{\pi_D}$) to a Poisson operator (bracket) on the submanifold \mathcal{S} (i.e. a symplectic leaf $\varphi_1 = \dots = \varphi_m = 0$ of π_D) in a standard way (reduction to a symplectic leaf). Namely, for arbitrary functions $f, g : \mathcal{S} \rightarrow \mathbb{R}$ one defines the *reduced Dirac bracket* $\{f, g\}_R$ on \mathcal{S} (with the corresponding Poisson operator π_R) as the restriction of the bracket of arbitrary prolongations of f and g to \mathcal{M} , i.e.

$$\{f, g\}_R = \{F, G\}_D|_{\mathcal{S}}$$

where $F, G : \mathcal{M} \rightarrow \mathbb{R}$ are two functions such that $f = F|_{\mathcal{S}}$ and $g = G|_{\mathcal{S}}$. Of course, an identical construction can be induced on any leaf \mathcal{F}_s from the foliation \mathcal{F} . There arises, of course, a question how 'robust' this construction is, i.e. to what extent the obtained deformation π_D and the reduction π_R are

independent of the choice of particular functions φ_i that define our submanifold S . This issue will be partially addressed in next sections.

The results of this section as well as the results presented in Sections 4 and 5 suggest that the concept of the classification of constraints as being either of "first-class" or of "second-class", proposed by Dirac, should be reexamined when one looks on the problem from the point of view of Poisson geometry. First of all it is clear that the procedure called *Dirac reduction* has two different levels. The first level we call *Dirac deformation* as we deform a Poisson bivector π from manifold \mathcal{M} to another Poisson bivector π_D on the same manifold \mathcal{M} . A sufficient condition for the existence of π_D for a given set of constraints φ_i , $i = 1, \dots, m$ is a nondegeneracy of the Gram matrix φ . From the construction all constraints φ_i are Casimirs of π_D . The second level of the construction we call *Dirac restriction* as we restrict a Poisson bivector π_D to its symplectic leaf \mathcal{F}_s . In the original Dirac construction it was a particular one, namely $\mathcal{F}_0 = S$. A Poisson bivector on \mathcal{F}_s is denoted by π_R . For the existence of the second step additional restrictions on φ_i have to be imposed. Actually, π_R has to be nonsingular. In a standard classification it means that we have to exclude constraints of first-class. Let us remind that a constraint φ_k is of *first class* if its Poisson bracket with all the remaining constants φ_i vanishes on S , that is if

$$\{\varphi_k, \varphi_i\}_\pi|_S = 0, \quad i = 1, \dots, m. \quad (10)$$

Otherwise φ_k is of *second-class*. In general, first-class constraints make π_R singular so that the Dirac reduction procedure can not be performed. Nevertheless, the condition (10) seems to be too strong. In sections 4 and 5 we demonstrate situations where constraints are of first-class but the singularity is 'removable' and so the Poisson bivector π_R is well defined. It suggests that the classification of constraints given by Dirac should be reformulated in the context of Poisson pencils and Poisson geometry.

3 Separable quasi-bi-Hamiltonian chains of Benenti type

In the following section we briefly remind basic facts about separable Hamiltonian systems on Riemannian manifolds, which form a special class of quasi-bi-Hamiltonian chains [3],[11], known also as the so called Benenti systems [21],[22]. Let (Q, g) be a Riemannian manifold with covariant metric tensor $g = (g_{ij})$ and with the inverse (contravariant metric tensor) $g^{-1} = G = (G^{ij})$. Let (q^1, \dots, q^n) be some coordinate system on Q and let $(q^1, \dots, q^n, p_1, \dots, p_n)$ be the corresponding canonical coordinates on the phase space $\mathcal{N} = T^*Q$ with the associated Poisson tensor

$$\theta_0 = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} \quad (11)$$

where I_n is $n \times n$ unit matrix and 0_n is the $n \times n$ matrix with all entries equal to zero. Let us consider the Hamiltonian $E : \mathcal{N} \rightarrow \mathbb{R}$ for the geodesic flow on

Q :

$$E = E(q, p) = \sum_{i,j=1}^n G^{ij}(q) p_i p_j. \quad (12)$$

As it is known, a $(1,1)$ -type tensor $B = (B_j^i)$ (or a $(2,0)$ -type tensor $B = (B^{ij})$) is called a Killing tensor with respect to g if $\{\sum (BG)^{ij} p_i p_j, E\}_{\theta_0} = 0$ (or $\{\sum (B)^{ij} p_i p_j, E\}_{\theta_0} = 0$). An important generalization of this notion is formulated in the following definition.

Definition 4 Let $L = (L_j^i)$ be a second order mixed type (i.e. $(1,1)$ -) tensor on Q and let $\bar{L} : \mathcal{N} \rightarrow \mathbb{R}$ be a function on \mathcal{N} defined as $\bar{L} = \frac{1}{2} \sum_{i,j=1}^n (LG)^{ij} p_i p_j$, where LG is a $(1,1)$ tensor with components $(LG)^{ij} = \sum_{k=1}^n L_k^i g^{kj}$. If

$$\{\bar{L}, E\}_{\theta_0} = \alpha E, \quad \text{where} \quad \alpha = \sum_{i,j=1}^n G^{ij} \frac{\partial f}{\partial q^i} p_j, \quad f = \text{Tr}(L),$$

then L is called a special conformal Killing tensor with the associated potential $f = \text{Tr}(L)$ [11].

The importance of this notion lies in the fact that on manifolds with tensor L the geodesic flows are separable. There exist, in this case, n constants of motion, quadratic in momenta, of the form

$$E_r = \sum_{i,j=1}^n A_r^{ij} p_i p_j = \sum_{i,j=1}^n (K_r G)^{ij} p_i p_j, \quad r = 1, \dots, n, \quad (13)$$

$$(K_r G)^{ij} = \sum_{k=1}^n (K_r)_k^i G^{kj}$$

where A_r and K_r are Killing tensors of type $(2,0)$ and $(1,1)$, respectively. Moreover, all the Killing tensors K_r are given by the following 'cofactor' formula

$$\text{cof}(\xi I - L) = \sum_{i=1}^{n-1} K_{n-i} \xi^i, \quad (14)$$

where $\text{cof}(A)$ stands for the matrix of cofactors, so that $\text{cof}(A)A = (\det A)I$. Notice that $K_1 = I$, hence $A_1 = G$ and $E_1 \equiv E$. Since the tensors K_r are Killing, with a common set of eigenfunctions, the functions E_r satisfy $\{E_s, E_r\}_{\theta_0} = 0$ and thus they constitute a system of n constants of motion in involution with respect to the Poisson structure θ_0 . So, for a given metric tensor g , the existence of a special conformal Killing tensor L is a sufficient condition for the geodesic flow on \mathcal{N} to be a Liouville integrable Hamiltonian system.

The special conformal Killing tensor L can be lifted from Q to a $(1,1)$ -type tensor on $\mathcal{N} = T^*Q$ where it takes the form

$$N = \begin{pmatrix} L & 0_n \\ F & L^T \end{pmatrix}, \quad F_j^i = \frac{\partial}{\partial q^i}(Lp)_j - \frac{\partial}{\partial q^j}(p^T L)_i. \quad (15)$$

The lifted $(1,1)$ tensor N is called a *recursion operator*. An important property of N is that when it acts on the canonical Poisson tensor θ_0 it produces another Poisson tensor

$$\theta_1 = N\theta_0 = \begin{pmatrix} 0 & L \\ -L^T & F \end{pmatrix},$$

compatible with the canonical one (actually θ_0 is compatible with $N^k\theta_0$ for any integer k).

It is now possible to show that the geodesic Hamiltonians E_r satisfy on $\mathcal{N} = T^*Q$ the set of relations

$$\theta_0 dE_{r+1} = \theta_1 dE_r + \rho_r \theta_0 dE_1, \quad E_{n+1} = 0, \quad r = 1, \dots, n, \quad (16)$$

where the functions $\rho_r(q)$ are coefficients of the characteristic polynomial of L (i.e. minimal polynomial of N), which is a special case of the quasi-bi-Hamiltonian chain (3) [1].

It turns out that with the tensor L we can (generically) associate a coordinate system on \mathcal{N} in which the flows associated with all the functions E_r separate. Namely, let $(\lambda^1(q), \dots, \lambda^n(q))$ be n distinct, functionally independent eigenvalues of L , i.e. solutions of the characteristic equation $\det(\xi I - L) = 0$. Solving these relations with respect to q we get the transformation $\lambda \rightarrow q$

$$q^i = \alpha_i(\lambda), \quad i = 1, \dots, n. \quad (17)$$

The remaining part of the transformation to the separation coordinates can be obtained as a canonical transformation reconstructed from the generating function $W(p, \lambda) = \sum_i p_i \alpha_i(\lambda)$ in the standard way by solving the implicit relations $\mu_i = \frac{\partial W(p, \lambda)}{\partial \lambda^i}$ with respect to p_i obtaining $p_i = \beta_i(\lambda, \mu)$. In the (λ, μ) coordinates, known as the Darboux-Nijenhuis coordinates (DN), the tensor L is diagonal $L = \text{diag}(\lambda^1, \dots, \lambda^n) \equiv \Lambda_n$, while the Hamiltonians (13) of our quasi-bi-Hamiltonian chain attain the form [3]

$$E_r(\lambda, \mu) = - \sum_{i=1}^n \frac{\partial \rho_r}{\partial \lambda^i} \frac{f_i(\lambda^i) \mu_i^2}{\Delta_i} \quad r = 1, \dots, n,$$

where

$$\Delta_i = \prod_{k=1, \dots, n, k \neq i} (\lambda^i - \lambda^k),$$

$\rho_r(\lambda)$ are symmetric polynomials (Viéte polynomials) defined by the relation

$$\det(\xi I - \Lambda) = (\xi - \lambda^1)(\xi - \lambda^2) \dots (\xi - \lambda^n) = \sum_{r=0}^n \rho_r \xi^r, \quad (18)$$

and where f_i are arbitrary smooth functions of one real argument.

It turns out that there exists a sequence of generic separable potentials $V_r^{(k)}$, $k \in \mathbf{Z}$, which can be added to geodesic Hamiltonians E_r such that the new Hamiltonians

$$H_r(q, p) = E_r(q, p) + V_r^{(k)}(q), \quad r = 1, \dots, n, \quad (19)$$

are still separable in the same coordinates (λ, μ) . These generic potentials are given by some recursion relations [4],[10]. The Hamiltonians $H_r : \mathcal{N} \rightarrow \mathbb{R}$ in (19) satisfy the following quasi-bi-Hamiltonian chain

$$\theta_0 dH_{r+1} = \theta_1 dH_r + \rho_r \theta_0 dH_1, \quad H_{n+1} = 0, \quad r = 1, \dots, n. \quad (20)$$

In the DN coordinates the Hamiltonians H_r attain the form [3]

$$H_r(\lambda, \mu) = - \sum_{i=1}^n \frac{\partial \rho_r}{\partial \lambda^i} \frac{f_i(\lambda^i) \mu_i^2 + \gamma_i(\lambda^i)}{\Delta_i}, \quad r = 1, \dots, n, \quad (21)$$

where potentials $V_r^{(k)}$ enter H_r as $\gamma_i(\lambda^i) = (\lambda^i)^{n+k-1}$. From (21) it immediately follows that in (λ, μ) variables the contravariant metric tensor G and all the Killing tensors K_r are diagonal

$$G^{ij} = \frac{f_i(\lambda^i)}{\Delta_i} \delta^{ij}, \quad (K_r)^i_j = - \frac{\partial \rho_r}{\partial \lambda^i} \delta^i_j.$$

Moreover, in the (λ, μ) coordinates the recursion operator and the tensor θ_1 attain the form

$$N = \begin{pmatrix} \Lambda_n & 0_n \\ 0_n & \Lambda_n \end{pmatrix}, \quad \theta_1 = \begin{pmatrix} 0_n & \Lambda_n \\ -\Lambda_n & 0_n \end{pmatrix}$$

while θ_0 remains in the form (11) since the transformation $(q, p) \rightarrow (\lambda, \mu)$ is canonical.

The quasi-bi-Hamiltonian chain (20) on \mathcal{N} can easily be lifted to a bi-Hamiltonian chain (2) on the extended manifold $\mathcal{M} = T^*Q \times \mathbf{R} = \mathcal{N} \times \mathbf{R}$ [11].

Having such a complete picture of separable quasi-bi-Hamiltonian chains on Riemannian manifolds, one can ask a question: what kind of holonomic constraints can be imposed on considered systems so that this quasi-bi-Hamiltonian separability schema is preserved? The simplest admissible case of such constraints will be considered in the next sections.

4 Reduction $\lambda^n = 0$

Let us consider a particle moving in our Riemannian manifold Q equipped with the coordinates (q^1, \dots, q^n) . Suppose that this particle is subordinated to some holonomic constraints on Q defined by the set of relations

$$\varphi_k(q) = 0, \quad k = 1, \dots, r \quad (22)$$

that define some submanifold of Q . The velocity $v = \sum_i v^i \frac{\partial}{\partial q^i}$ of this particle must then remain tangent to this submanifold so that

$$0 = \langle d\varphi_k, v \rangle = \sum_{i=1}^n \frac{\partial \varphi_k}{\partial q^i} v^i.$$

and thus in our coordinates $v^i = \sum_j G^{ij} p_j$ the motion of the particle in the phase space $\mathcal{N} = T^*Q$ is constrained not only by the r relations (22) but also by the r relations

$$\varphi_{r+k}(q, p) \equiv \sum_{i,j=1}^n G^{ij}(q) \frac{\partial \varphi_k(q)}{\partial q^i} p_j = 0, \quad k = 1, \dots, r. \quad (23)$$

that are nothing else than the lift of (22) to \mathcal{N} . We will call the constraints (23) a *g-consequence* of the constraints (22), as they are natural differential consequences of (22) at given metric tensor g . The constraints (22)-(23) define a submanifold \mathcal{S} of \mathcal{N} of dimension $n - 2r$.

Let us now consider our quasi-bi-Hamiltonian chain (20) in \mathcal{N} . We would like to know what types of holonomic constraints (22) on Q do not destroy the separability of the constrained chain? This is a complicated question and due to its nature it is most convenient (for a moment) to consider it directly in our separation coordinates (λ, μ) . Thus, in this section we will analyze only a very special choice of the functions φ_k in (22). Namely, we put $r = 1$ (so that $n - 2r = n - 2$ which corresponds to $m = 2$ in Section 2) and define the corresponding function φ'_1 in (λ, μ) variables as

$$\varphi'_1(\lambda) = \lambda^n \quad (24)$$

Since the metric tensor $G = (G^{ij})$ has in (λ, μ) coordinates the diagonal form $G = \text{diag} (f_i(\lambda^i)/\Delta_i)$, $i = 1, \dots, n$ and since the equation (23) has an invariant form (i.e. in (λ, μ) coordinates it has the same form with q and p replaced by λ and μ respectively) the *g-consequence* of (24) reads as

$$\varphi'_2(\lambda, \mu) = \frac{f_n(\lambda^n)}{\Delta_n} \mu_n \quad (25)$$

(we use ' here since soon we will modify the constraints (24)-(25) to a simpler form). These two constraints define a subset $\mathcal{S}' = \{\varphi'_1 = 0, \varphi'_2 = 0\}$ of \mathcal{N} . We will however restrict us to a submanifold $\mathcal{S} = \{\lambda^n = 0, \mu_n = 0\}$ neglecting the term $f_n(\lambda^n)/\Delta_n$ in φ'_2 (for $\Delta_n \neq 0$ $\mathcal{S} \subset \mathcal{S}'$ and in case of the systems when f_n never vanishes \mathcal{S} and \mathcal{S}' coincide). The reason for this is that various calculations with the help of these new constraints $\lambda^n = 0, \mu_n = 0$ are simpler. Let us now perform the Dirac reduction of our Poisson operators θ_0 and θ_1

$$\theta_0 = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}, \quad \theta_1 = \begin{pmatrix} 0_n & \Lambda_n \\ -\Lambda_n & 0_n \end{pmatrix} \quad (26)$$

and of the corresponding quasi-bi-Hamiltonian chain (16) (geodesic case) or (20) (potential case) on the submanifold \mathcal{S} . We can do it either by using the constraints (24)-(25) or by the constraints that directly describe \mathcal{S} :

$$\varphi_1(\lambda) \equiv \lambda^n = 0 \quad , \quad \varphi_2(\lambda, \mu) \equiv \mu_n = 0 \quad (27)$$

(of course $\varphi_1 \equiv \varphi'_1$). It turns out that the corresponding Dirac deformations (given by (8) or equivalently by (9)) $\theta_{0,D}$ and $\theta'_{0,D}$ of θ_0 are *not* equal. Similarly, $\theta_{1,D} \neq \theta'_{1,D}$. However, after reducing $\theta_{0,D}$ and $\theta'_{0,D}$ on \mathcal{S} we obtain the same Poisson operator (and similarly with $\theta_{1,D}$ and $\theta'_{1,D}$).

Theorem 5 *With the notation as above, the reduced operators $\theta_{i,R}$ and $\theta'_{i,R}$ on \mathcal{S} satisfy $\theta'_{0,R} = \theta_{0,R}$ and $\theta'_{1,R} = \theta_{1,R}$. If we parametrize \mathcal{S} by the variables $(\lambda^1, \dots, \lambda^{n-1}, \mu_1, \dots, \mu_{n-1})$ then*

$$\theta_{0,R} = \begin{pmatrix} 0_{n-1} & I_{n-1} \\ -I_{n-1} & 0_{n-1} \end{pmatrix} \quad , \quad \theta_{1,R} = \begin{bmatrix} 0_{n-1} & \Lambda_{n-1} \\ -\Lambda_{n-1} & 0_{n-1} \end{bmatrix} \quad (28)$$

with $\Lambda_{n-1} = \text{diag}(\lambda^1, \dots, \lambda^{n-1})$.

Proof. Let us start by calculating $\theta_{0,D}$. In this case the 2×2 Gram matrix $\varphi = (\{\varphi_i, \varphi_j\}_{\theta_0})$ is of the form

$$\varphi = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and thus according to (7) we have $Z_1 = X_2$ and $Z_2 = -X_1$. The formula (8) yields

$$\theta_{0,D} = \theta_0 - X_1 \wedge X_2 = \theta_0 - \frac{\partial}{\partial \lambda^n} \wedge \frac{\partial}{\partial \mu_n} = \sum_{i=1}^{n-1} \frac{\partial}{\partial \lambda^i} \wedge \frac{\partial}{\partial \mu_i},$$

since $X_1 = \theta_0 d\varphi_1 = -\partial/\partial \mu_n$ and $X_2 = \theta_0 d\varphi_2 = \partial/\partial \lambda^n$. So, in (λ, μ) coordinates we have

$$\theta_{0,D} = \left[\begin{array}{cc|cc} 0_n & & I_{n-1} & 0_{(n-1) \times 1} \\ & & 0_{1 \times (n-1)} & 0 \\ \hline -I_{n-1} & 0_{(n-1) \times 1} & & \\ 0_{1 \times (n-1)} & 0 & & 0_n \end{array} \right] \quad (29)$$

The operator (29) has thus the n -th and the last column and the n -th and the last row filled with zeros. These two rows and two columns of zeros correspond to the fact (see Theorem 1) that the constraints $\varphi_1 = \lambda^n$ and $\varphi_2 = \mu_n$ are now Casimir functions for $\theta_{0,D}$. We can thus directly project $\theta_{0,D}$ onto the symplectic leaf $\mathcal{S} = \{\lambda^n = \mu_n = 0\}$ of $\theta_{0,D}$ by simply removing the two zero columns and zero rows from (29), which yield the operator $\theta_{0,R}$ that written in coordinates $(\lambda^1, \dots, \lambda^{n-1}, \mu_1, \dots, \mu_{n-1})$ on \mathcal{S} attains the form as in (28). Similar computations show that in the case of $\theta_{1,D}$

$$\varphi = \begin{bmatrix} 0 & \lambda^n \\ -\lambda^n & 0 \end{bmatrix} = \begin{bmatrix} 0 & \varphi_1 \\ -\varphi_1 & 0 \end{bmatrix}$$

and thus (see (10)) φ_1 and φ_2 are first-class constraints. Indeed, following (7) we have $Z_1 = \frac{1}{\lambda^n} X_2$ and $Z_2 = -\frac{1}{\lambda^n} X_1$ and the formula (8) yields

$$\theta_{1,D} = \theta_1 - \frac{1}{\lambda^n} X_1 \wedge X_2$$

which seems to be singular on \mathcal{S} . Nevertheless this singularity is 'removable' since $X_1 = \theta_1 d\varphi_1 = -\lambda^n \partial/\partial \mu_n$, $X_2 = \theta_1 d\varphi_2 = \lambda^n \partial/\partial \lambda^n$ and hence

$$\theta_{1,D} = \theta_1 - \lambda^n \frac{\partial}{\partial \lambda^n} \wedge \frac{\partial}{\partial \mu_n} = \sum_{i=1}^{n-1} \lambda^i \frac{\partial}{\partial \lambda^i} \wedge \frac{\partial}{\partial \mu_i}.$$

So, the matrix form of $\theta_{1,D}$ in (λ, μ) coordinates becomes

$$\theta_{1,D} = \left[\begin{array}{cc|cc} & 0_n & \Lambda_{n-1} & 0_{(n-1) \times 1} \\ & & 0_{1 \times (n-1)} & 0 \\ \hline -\Lambda_{n-1} & 0_{(n-1) \times 1} & & \\ 0_{1 \times (n-1)} & 0 & & 0_n \end{array} \right]$$

and the projection of this operator onto the symplectic leaf $\mathcal{S} = \{\lambda^n = \mu_n = 0\}$ of $\theta_{1,D}$ yields exactly $\theta_{1,R}$ as in (28).

Passing to φ'_1, φ'_2 notice that

$$\varphi'_{12} = \{\varphi'_1, \varphi'_2\}_{\theta_0} = \frac{f_n(\lambda^n)}{\Delta_n},$$

so one can have a situation when $\varphi'_{12}|_{\mathcal{S}} = 0$ which again leads to first-class constraints according to Dirac classification. Nevertheless, calculations similar to these for φ_1, φ_2 show that singularities are again 'removable' and we end up with the same form of $\theta_{0,D}$ and $\theta_{1,D}$. ■

Of course, the reduced operators $\theta_{0,R}$ and $\theta_{1,R}$ are compatible. Moreover, it is easy to see that the Hamiltonians (21) restricted to \mathcal{S} become

$$\begin{aligned} & H_{r,R}(\lambda^1, \dots, \lambda^{n-1}, \mu_1, \dots, \mu_{n-1}) \\ &= H_r|_{\mathcal{S}} = - \sum_{i=1}^{n-1} \frac{\partial \rho_{r,R}}{\partial \lambda^i} \frac{f_{i,R}(\lambda^i) \mu_i^2 + \gamma_{i,R}(\lambda^i)}{\Delta_{i,R}}, \quad r = 1, \dots, n-1, \end{aligned}$$

$$H_{n,R} = H_n|_{\mathcal{S}} = 0$$

where $f_{i,R}(\lambda^i) = f_i(\lambda^i)/\lambda^i$, $\gamma_{i,R}(\lambda^i) = \gamma_i(\lambda^i)/\lambda^i$, $i = 1, \dots, n-1$ and $\rho_{r,R}(\lambda)$ are Viète polynomials of dimension $n-1$. Having all this in mind, we can formulate the following corollary.

Corollary 6 *Both Dirac reductions (24)-(25) and (27) of the quasi-bi-Hamiltonian system (20) onto \mathcal{S} lead to the same quasi-bi-Hamiltonian system of the form*

$$\theta_{0,R} dH_{r+1,R} = \theta_{1,R} dH_{r,R} + \rho_{r,R} \theta_{0,R} dH_{1,R}, \quad H_{n,R} = 0, \quad r = 1, \dots, n-1.$$

which is thus separable in the variables $(\lambda^1, \dots, \lambda^{n-1}, \mu_1, \dots, \mu_{n-1})$ that coincide on \mathcal{S} with 'first' $n-2$ separation coordinates from our coordinate system (λ, μ) that separates (20).

Because the reduced quasi-bi-Hamiltonian chain has exactly the form (20), it can be put into a bi-Hamiltonian chain on $\mathcal{S} \times \mathbb{R}$.

Of course this procedure of Dirac reduction, in principle could be performed directly in (q, p) coordinates, since the formula (8) has a tensor character and thus yields the same result no matter what coordinate system we choose for actual calculating of deformations $\theta_{0,D}$ and $\theta_{1,D}$. There is however a major obstacle here: we are usually not able to express the constraint $\lambda^n = 0$ in 'physical' coordinates (q, p) , as equations (17) are noninvertible in general, i.e. there is no algebraic way of solving them with respect to λ (if we could, then the second constraint (25) could be computed as the g -consequence of the first one calculated directly in (q, p) coordinates, with the help of (23)). Thus, although the picture presented in this section is clear, nevertheless it is somehow useless once we are given the quasi-bi-Hamiltonian chain in natural (q, p) coordinates. In the next section we will demonstrate how this problem can be defused by the use of the so called Hankel-Frobenius coordinates.

5 Reduction in Hankel-Frobenius coordinates

As it was demonstrated in the previous section, the constraints $\varphi_1 = \lambda^n$, $\varphi_2 = \mu_n$ preserve the separability on the constrained submanifold \mathcal{S} but are very inconvenient to handle with, as in general we do not know how to express them in the original coordinates (q, p) . More convenient for this purpose is the set of so called Hankel-Frobenius coordinates $(\rho_i, v_i)_{i=1}^n$ [24],[25]. They are non-canonical coordinates, related to the separated coordinates in the following way

$$\begin{aligned} \rho_i &= \rho_i(\lambda), \quad i = 1, \dots, n, \\ v_i &= \sum_j (V_n^{-1})_{ij} \mu_j = \sum_k \frac{\partial \rho_i}{\partial \lambda^k} \frac{1}{\Delta_k} \mu_k, \quad i = 1, \dots, n, \end{aligned} \quad (30)$$

where

$$V_n = \begin{pmatrix} (\lambda^1)^{n-1} & \dots & \lambda^1 & 1 \\ \vdots & & \vdots & \vdots \\ (\lambda^n)^{n-1} & \dots & \lambda^n & 1 \end{pmatrix}$$

is a Vandermonde matrix and ρ_i are Viéte polynomials (18). The mapping (30) is not a point transformation on Q and therefore it makes no sense to distinguish covariant and contravariant indices now. Applying the map (30) we obtain the form of operators θ_0 and θ_1 and the $2n$ -dimensional recursion operator N_n (15) in (ρ, v) coordinates:

$$\theta_0 = \begin{pmatrix} 0 & U_n \\ -U_n & 0 \end{pmatrix}, \quad U_n = \begin{pmatrix} 0 & 0 & \dots & 1 \\ 0 & \dots & 1 & \rho_1 \\ \vdots & \dots & \dots & \vdots \\ 1 & \rho_1 & \dots & \rho_{n-1} \end{pmatrix}, \quad (31)$$

$$\theta_1 = N_n \theta_0 = \begin{pmatrix} 0 & F_n U_n \\ -F_n U_n & 0 \end{pmatrix}, \quad F_n U_n = U_n F_n^T. \quad (32)$$

$$N_n = \begin{pmatrix} F_n & 0 \\ 0 & F_n \end{pmatrix}, \quad F_n = \begin{pmatrix} -\rho_1 & 1 & \cdots & \cdots & 0 \\ -\rho_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ -\rho_{n-1} & 0 & \cdots & \cdots & 1 \\ -\rho_n & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

We will now consider yet another deformation (8) of θ_0 and θ_1 , given in (λ, μ) variables by the constraints

$$\begin{aligned} \varphi_1'' &= \rho_n(\lambda) = (-1)^n \lambda^1 \lambda^2 \cdots \lambda^n \\ \varphi_2'' &= v_n(\lambda) = (-1)^n \sum_{j=1}^n \frac{\mu_j}{\Delta_j} \lambda^1 \lambda^2 \cdots \lambda^{j-1} \lambda^{j+1} \cdots \lambda^n \end{aligned} \quad (33)$$

equal to last pair of Hankel-Frobenius coordinates. The constraints (33) are related with the constraints (27) as

$$\begin{aligned} \varphi_1'' &= \psi_1 \varphi_1 \\ \varphi_2'' &= \psi_2 \varphi_1 + \psi_3 \varphi_2 \end{aligned} \quad (34)$$

where the functions

$$\begin{aligned} \psi_1 &= (-1)^n \lambda^1 \lambda^2 \cdots \lambda^{n-1} \\ \psi_2 &= (-1)^n \sum_{j=1}^{n-1} \frac{\mu_j}{\Delta_j} \lambda^1 \lambda^2 \cdots \lambda^{j-1} \lambda^{j+1} \cdots \lambda^{n-1} \end{aligned}$$

and $\psi_3 = (-1)^n \lambda^1 \lambda^2 \cdots \lambda^{n-1} / \Delta_n$ never vanish at $\mathcal{S} = \{\varphi_1 = \varphi_2 = 0\}$ (in fact, $\psi_2|_{\mathcal{S}} = -1$ and $\psi_3|_{\mathcal{S}} = -(\mu_1 + \cdots + \mu_{n-1})$) and thus the constraints (33) define (locally) the same submanifold \mathcal{S} as the constraints (27). The corresponding deformations $\theta_{0,D}''$ and $\theta_{1,D}''$ of θ_0 and θ_1 will of course not be equal to $\theta_{0,D}$ and $\theta_{1,D}$, but again it turns out that their reductions on \mathcal{S} will coincide with the corresponding reductions of $\theta_{0,D}$ and $\theta_{1,D}$ on \mathcal{S} .

Theorem 7 *In the notation as above, $\theta_{0,R}'' = \theta_{0,R}$ and $\theta_{1,R}'' = \theta_{1,R}$*

Proof. We will use (9) rather than (8) since it turns out that the calculations are in this case simpler when one uses bracket definition of Dirac deformation than bivector definition. Applying (9), we easily get that for any two functions $A, B : \mathcal{N} \rightarrow R$

$$\{A, B\}_{\theta_{0,D}} = \{A, B\}_{\theta_0} + \frac{\{A, \varphi_2\}_{\theta_0} \{B, \varphi_1\}_{\theta_0} - \{A, \varphi_1\}_{\theta_0} \{B, \varphi_2\}_{\theta_0}}{\{\varphi_1, \varphi_2\}_{\theta_0}}$$

where of course $\{\varphi_1, \varphi_2\}_{\theta_0} = 1$ and so it does not vanish on \mathcal{S} . Similarly

$$\{A, B\}_{\theta_{0,D}''} = \{A, B\}_{\theta_0} + \frac{\{A, \varphi_2''\}_{\theta_0} \{B, \varphi_1''\}_{\theta_0} - \{A, \varphi_1''\}_{\theta_0} \{B, \varphi_2''\}_{\theta_0}}{\{\varphi_1'', \varphi_2''\}_{\theta_0}}. \quad (35)$$

Using the relations (34) between the deformed constraints φ_i'' and the original constraints φ_i , the Leibniz property of Poisson brackets and the fact that ψ_1 and ψ_3 depend only on λ , we obtain

$$\begin{aligned}\{\varphi_1'', \varphi_2''\}_{\theta_0} &= \psi_1 \varphi_1 \{\varphi_1, \varphi_2\}_{\theta_0} + \varphi_1^2 \{\psi_1, \varphi_2\}_{\theta_0} + \\ &\quad + \psi_1 \psi_3 \{\varphi_1, \varphi_2\}_{\theta_0} + \varphi_1 \varphi_2 \{\psi_1, \psi_3\}_{\theta_0},\end{aligned}$$

so that $\{\varphi_1'', \varphi_2''\}_{\theta_0}|_{\mathcal{S}} = \psi_1 \psi_3 \{\varphi_1, \varphi_2\}_{\theta_0}|_{\mathcal{S}} = \psi_1 \psi_3|_{\mathcal{S}}$. Similar calculations show that

$$\begin{aligned}&(\{A, \varphi_2''\}_{\theta_0} \{B, \varphi_1''\}_{\theta_0} - \{A, \varphi_1''\}_{\theta_0} \{B, \varphi_2''\}_{\theta_0})|_{\mathcal{S}} \\ &= \psi_1 \psi_3 (\{A, \varphi_2\}_{\theta_0} \{B, \varphi_1\}_{\theta_0} - \{A, \varphi_1\}_{\theta_0} \{B, \varphi_2\}_{\theta_0})|_{\mathcal{S}}\end{aligned}$$

and thus the factors $\psi_1 \psi_3$ in the numerator and in the denominator of (35) cancel and we conclude that

$$\{A, B\}_{\theta_{0,D}}|_{\mathcal{S}} = \{A, B\}_{\theta_{0,D}}|_{\mathcal{S}}$$

which is the same as to claim that $\theta_{0,R}'' = \theta_{0,R}$. The proof that $\theta_{1,R}'' = \theta_{1,R}$ is similar: first one shows that

$$\{\varphi_1'', \varphi_2''\}_{\theta_1}|_{\mathcal{S}} = \psi_1 \psi_3 \{\varphi_1, \varphi_2\}_{\theta_1}|_{\mathcal{S}} = \psi_1 \psi_3|_{\mathcal{S}} \lambda^n$$

which is by the way equal to zero on \mathcal{S} . However, we also get

$$\begin{aligned}&(\{A, \varphi_2''\}_{\theta_1} \{B, \varphi_1''\}_{\theta_1} - \{A, \varphi_1''\}_{\theta_1} \{B, \varphi_2''\}_{\theta_1})|_{\mathcal{S}} \\ &= \psi_1 \psi_3 (\{A, \varphi_2\}_{\theta_1} \{B, \varphi_1\}_{\theta_1} - \{A, \varphi_1\}_{\theta_1} \{B, \varphi_2\}_{\theta_1})|_{\mathcal{S}}\end{aligned}$$

so that on \mathcal{S}

$$\begin{aligned}\{A, B\}_{\theta_{1,D}}''|_{\mathcal{S}} &= \{A, B\}_{\theta_1}|_{\mathcal{S}} + \frac{\psi_1 \psi_3 (\{A, \varphi_2\}_{\theta_1} \{B, \varphi_1\}_{\theta_1} - \{A, \varphi_1\}_{\theta_1} \{B, \varphi_2\}_{\theta_1})}{\psi_1 \psi_3 \{\varphi_1, \varphi_2\}_{\theta_1}}|_{\mathcal{S}} \\ &= \{A, B\}_{\theta_1}|_{\mathcal{S}} + \frac{\{A, \varphi_2\}_{\theta_1} \{B, \varphi_1\}_{\theta_1} - \{A, \varphi_1\}_{\theta_1} \{B, \varphi_2\}_{\theta_1}}{\lambda^n}|_{\mathcal{S}} = \{A, B\}_{\theta_{1,D}}|_{\mathcal{S}}\end{aligned}$$

since the term λ^n in the last expression does not cause any singularity: for every possible combination $A, B = \lambda^i, \mu_j$ the numerator in the above expression is either zero or some multiple of λ^n . This proves that $\theta_{1,R}'' = \theta_{1,R}$. ■

Before we proceed with the main theme of this article, let us make a digression: we will establish the form of $\theta_{0,R}$ and $\theta_{1,R}$ in (ρ, v) -coordinates.

Lemma 8 *In (ρ, v) -coordinates we have*

$$\begin{aligned}\theta_{0,R} &= (N_{n-1})^{-1} \begin{pmatrix} 0 & U_{n-1} \\ -U_{n-1} & 0 \end{pmatrix} = \begin{pmatrix} 0_{n-1} & (F_{n-1})^{-1} U_{n-1} \\ -(F_{n-1})^{-1} U_{n-1} & 0_{n-1} \end{pmatrix} \\ \theta_{1,R} &= \begin{pmatrix} 0 & U_{n-1} \\ -U_{n-1} & 0 \end{pmatrix}\end{aligned}$$

This lemma can be proved either by direct calculating of the deformation formula (8) in (ρ, v) -coordinates or by transforming both $\theta_{0,R}$ and $\theta_{1,R}$ with the map that is the restriction of the map (30) to \mathcal{S} . This restricted map has no longer the form (30) and this is why the operators $\theta_{0,R}$ and $\theta_{1,R}$ does not transform respectively to the operators of the form (31) and (32) with n replaced by $n - 1$.

Having established Theorem 7 we can now, for a given (quasi)-bi-Hamiltonian system (20), perform the Dirac reduction $\lambda^n = \mu_n = 0$ directly in physical coordinates (q, p) by performing the equivalent reduction $\rho_n = v_n = 0$ provided that we are able to express the constraints (33) directly in physical coordinates. The first of the constraints in (33), i.e. $\rho_n = 0$ is given directly in (q, p) coordinates once the system (16) has been given (this is the reason for which we consider this constraint and Hankel-Frobenius coordinates). Below we shall show how to express the second constraint ($v_n = 0$) in physical coordinates.

Let us first observe that in case of systems for which $f_i(\lambda^i) = \text{const}$ (that does not depend on i) the constraint $v_n = 0$ calculated in (λ, μ) coordinates is just the g -consequence of the first one ($\rho_n = 0$), so that in this particular case we can easily get the function $\phi_2''(q, p)$ by calculating the expression (23) with an appropriate metric tensor g that can be found in h_1 . When the functions f_i are more complicated, we must proceed differently.

Theorem 9 *Assume that $f_i(\lambda^i) = f(\lambda^i)$, i.e. that the functions f_i do not depend on i . Let also F be the matrix given by (32). Then the coordinates v_i expressed in (q, p) coordinates have the following form*

$$v_i(q, p) = \sum_{j=1}^n \left((f(F)^{-1})_{ij} \sum_{k,l=1}^n G^{kl}(q) \frac{\partial \rho_j}{\partial q^k} p_l \right) \quad (36)$$

with F being expressed in q .

Proof. Let us expand the function f in a formal Laurent series

$$f(\lambda) = \sum_m a_m \lambda^m, \quad a_m \in \mathbb{R}, \quad m \in \mathbf{Z}_f \subset \mathbf{Z}. \quad (37)$$

Let also v'_k , $k = 1, \dots, n$ be the g -consequence of ρ_k so that

$$v'_k(\lambda, \mu) = \sum_{i=1}^n \frac{f_i}{\Delta_i} \frac{\partial \rho_k}{\partial \lambda^i} \mu_i \quad \text{and} \quad v'_k(q, p) = \sum_{i,j=1}^n G^{ij}(q) \frac{\partial \rho_k}{\partial q^i} p_j. \quad (38)$$

Setting $\mu = (\mu_1, \dots, \mu_n)^T$, $v = (v_1, \dots, v_n)^T$, $v' = (v'_1, \dots, v'_n)^T$ and using the fact (30) that $\mu = Vv$ (where V is the Vandermonde matrix) we easily obtain that

$$v' = V^{-1} \text{diag}(f(\lambda^1), \dots, f(\lambda^n)) V v$$

Moreover, from the transformation between (λ, μ) and (ρ, v) coordinates one finds that

$$V^{-1} \Lambda^m V = U(F^T)^m U^{-1} = F^m.$$

so that due to (37)

$$\begin{aligned} v' &= V^{-1} \sum_m a_m \operatorname{diag}((\lambda^1)^m, \dots, (\lambda^n)^m) V v = V^{-1} \left(\sum_m a_m \Lambda^m \right) V v \\ &= \left(\sum_m a_m V^{-1} \Lambda^m V \right) v = \left(\sum_m a_m F^m \right) v = f(F) v. \end{aligned}$$

This relation is valid in every coordinate system and thus $v(q, p) = f(F(q))v'(q, p)$ which due to (38) yields (36). ■

In case when the assumption of the theorem is not satisfied, we can not express v_n in (q, p) variables using the method presented in the proof. Since in practice this situation is very rare, we choose not to discuss it in this article. Notice that if $f = \text{const}$ then $f(F) = \text{const } I$ and the formula (36) for v_n reduces to the g -consequence of ρ_n , as it should be.

Remark 10 *One can ask the question why, in the case of $f \neq \text{const}$, instead of taking $\varphi_2'' = \nu_n$ as above, we do not choose φ_2'' to be simply the g -consequence of $\varphi_1' = \rho_n$, as both pairs of constraints describe the same \mathcal{S} . The answer is due to one of the fundamental observations of presented paper, mentioned at the end of section 2. Actually, a constrained submanifold \mathcal{S} can be defined by infinitely many different pairs of constraints, all of them of the first class according to the classical Dirac classification. That is, either the Gram matrix is singular: $\{\varphi_1', \varphi_2''\}_\theta = 0$, or $\{\varphi_1', \varphi_2''\}_\theta|_{\mathcal{S}} = 0$, for one or both θ_i , hence, in principle, there is no Dirac deformation $\theta_{i,D}$ or the restriction of $\theta_{i,D}$ on \mathcal{S} is not possible. However, among all these pairs of constraints, there are exceptional pairs, like (λ^n, μ_n) or (ρ_n, ν_n) , for which singularities are 'removable' and Dirac restricted Poisson pencil is nonsingular. So in fact, these particular constraints have to be considered as second order constraints.*

Notice also that in order to determine the constraint v_n from (36) we have first to find the function f for example from E_1 , written in (λ, μ) coordinates. So in practice, we cannot avoid the calculation of separation coordinates, and this is the price we have to pay. Nevertheless, in general, it is not difficult to find the transformation (17) and then the geodesic Hamiltonian $E_1(\lambda, \mu)$ for the original system. Also, one has to bear in mind that even though the reduction procedure looks trivial in (λ, μ) -coordinates, it is usually not possible to express the obtained reduced system back in (q, p) -coordinates since our Hamiltonians $E_{r,R}$ have different function f than the Hamiltonians E_r . That is why we had to use Hankel-Frobenius coordinates.

6 Examples

In this chapter we will illustrate the introduced ideas by two examples. In our first example of Dirac reductions of separable systems we will consider the so called first Newton representation of the seventh-order stationary flow of

the KdV hierarchy [26],[2]. It is a Lagrangian system of second order Newton equations

$$\begin{aligned} q_{,tt}^1 &= -10(q^1)^2 + 4q^2 \\ q_{,tt}^2 &= -16q^1q^2 + 10(q^1)^3 + 4q^3 \\ q_{,tt}^3 &= -20q^1q^3 - 8(q^2)^2 + 30(q^1)^2q^2 - 15(q^1)^4 + c \end{aligned}$$

(where the subscript $,t$ denotes the differentiation with respect to the evolution parameter $t \in \mathbb{R}$). The above system can be represented as a quasi-bi-Hamiltonian system on \mathcal{N} belonging to a quasi-bi-Hamiltonian chain of the form (20) with $n = 3$, with Hamiltonians

$$H_1 = p_1p_3 + \frac{1}{2}p_2^2 + 10(q^1)^2q^3 - 4q^2q^3 + 8q^1(q^2)^2 - 10(q^1)^3q^2 + 3(q^1)^5 \quad (39)$$

$$\begin{aligned} H_2 &= \frac{1}{2}q^3p_3^2 - \frac{1}{2}q^1p_2^2 + \frac{1}{2}q^2p_2p_3 - \frac{1}{2}p_1p_2 - \frac{1}{2}q^1p_1p_3 + 2(q^1)^2(q^2)^2 + \frac{5}{2}(q^1)^4q^2 \\ &\quad - \frac{5}{4}(q^1)^6 - 2(q^2)^3 + (q^3)^2 - 6q^1q^2q^3 \end{aligned} \quad (40)$$

$$\begin{aligned} H_3 &= \frac{1}{8}(q^2)^2p_3^2 + \frac{1}{8}(q^1)^2p_2^2 + \frac{1}{8}p_1^2 + \frac{1}{4}q^1p_1p_2 + \frac{1}{4}q^2p_1p_3 - \frac{1}{4}q^1q^2p_2p_3 \\ &\quad - \frac{1}{2}q^3p_2p_3 - 3(q^1)^3(q^2)^2 + q^1(q^2)^3 + \frac{5}{4}(q^1)^5q^2 + 2q^1(q^3)^2 \\ &\quad + \frac{5}{4}(q^1)^4q^3 + (q^2)^2q^3 - (q^1)^2q^2q^3 \end{aligned}$$

with the corresponding canonical operator θ_0 (11) and θ_1 of the form

$$\theta_1 = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & q^1 & -1 & 0 \\ 0 & 0 & 0 & q^2 & 0 & -1 \\ 0 & 0 & 0 & 2q^3 & q^2 & q^1 \\ -q^1 & -q^2 & -2q^3 & 0 & p_2 & p_3 \\ 1 & 0 & -q^2 & -p_2 & 0 & 0 \\ 0 & 1 & -q^1 & -p_3 & 0 & 0 \end{bmatrix}.$$

and with $\rho_1 = -q^1$, $\rho_2 = \frac{1}{4}(q^1)^2 + \frac{1}{2}q^2$, $\rho_3 = -\frac{1}{4}q^1q^2 - \frac{1}{4}q^3$.

From the form of H_1 one can directly see that the inverse metric tensor G expressed in (q, p) variables has in this example an anti-diagonal form

$$G = \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad (41)$$

while the conformal Killing tensor L has the form

$$L = \frac{1}{2} \begin{bmatrix} q^1 & -1 & 0 \\ q^2 & 0 & -1 \\ 2q^3 & q^2 & q^1 \end{bmatrix}$$

which substituted in (14) yields the geodesic parts of all the Hamiltonians (40). Our quasi-bi-Hamiltonian system turns out to be separable in (λ, μ) coordinates

defined as above and it turns out that in this case $f_i = 1/8 = \text{const}$ and $\gamma_i(\lambda^i) = 16(\lambda^i)^7$ in (21). Thus, we can easily find out the constraints $\rho_3 = v_3 = 0$ directly in (q, p) coordinates. From the form of H_3 in (40) one can see that $\varphi_1'' = \rho_3(q) = -\frac{1}{4}(q_1 q_2 + q_3)$ and that $\varphi_2'' = v_3(q, p)$ is just the g -consequence of ρ_3 . An easy computation of (23) with the use of (41) yields that $v_3(q, p) = -\frac{1}{8}(p_1 + p_2 q^1 + p_3 q^2)$. Performing the Dirac deformation (8) of θ_0 and θ_1 with the use of constraints φ_1'' and φ_2'' yields

$$\theta_{0,D} = \frac{1}{2q^2 + (q^1)^2} \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & q^2 + (q^1)^2 & -q^1 & -1 \\ 0 & 0 & 0 & -q^1 q^2 & 2q^2 & -q^1 \\ 0 & 0 & 0 & -(q^2)^2 & -q^1 q^2 & q^2 + (q^1)^2 \\ \hline & & -*^T & 0 & q^1 p_2 - q^2 p_3 & p_2 \\ & & & q^2 p_3 - q^1 p_2 & 0 & p_3 \\ & & & -p_2 & -p_3 & 0 \end{array} \right]$$

and

$$\theta_{1,D} = \frac{1}{2} \left[\begin{array}{ccccc} 0 & 0 & 0 & q^1 & -1 & 0 \\ 0 & 0 & 0 & q^2 & 0 & -1 \\ 0 & 0 & 0 & -2q^1 q^2 & q^2 & q^1 \\ -q^1 & -q^2 & 2q^1 q^2 & 0 & p_2 & p_3 \\ 1 & 0 & -q^2 & -p_2 & 0 & 0 \\ 0 & 1 & -q^1 & -p_3 & 0 & 0 \end{array} \right]$$

so that $\theta_{1,D}$ differs from θ_1 only at entries (3,4) and (4,3), where q^3 was deformed to $-q^1 q^2$ (notice that on \mathcal{S} indeed we have $q^3 = -q^1 q^2$). We will now pass to the Casimir variables chosen as variables

$$(q^1, q^2, \varphi_1''(q), \varphi_2''(q, p), p_2, p_3) \quad (42)$$

since due to the fact that it is easiest to eliminate q^3 and p_1 from the system of equations $\varphi_1'' = \varphi_1''(q)$, $\varphi_2'' = \varphi_2''(q, p)$ we will parametrize our submanifold by the coordinates (q^1, q^2, p_2, p_3) . In the variables (42) the operators $\theta_{i,D}$ attain the form

$$\theta_{0,D} = \frac{1}{2q^2 + (q^1)^2} \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & -q^1 & -1 \\ 0 & 0 & 0 & 0 & 2q^2 & -q^1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ q^1 & 2q^2 & 0 & 0 & 0 & p_3 \\ 1 & q^1 & 0 & 0 & -p_3 & 0 \end{array} \right]$$

and

$$\theta_{1,D} = \frac{1}{2} \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array} \right]$$

respectively. Observe that in these new variables all the entries at third and fourth rows and columns are zero, as it should be, since now the Casimirs φ_1''

and φ_2'' are part of our coordinate system. We can now write down the reduced operators $\theta_{0,R}$ and $\theta_{1,R}$ in variables (q^1, q^2, p_2, p_3) by simply removing the zero rows and columns (we would also have to put $\varphi_i'' = 0$ but our matrices $\theta_{i,D}$ do not contain any variables φ_i'' in their entries). This yields

$$\theta_{0,R} = \frac{1}{2q^2 + (q^1)^2} \begin{bmatrix} 0 & 0 & -q^1 & -1 \\ 0 & 0 & 2q^2 & -q^1 \\ q^1 & -2q^2 & 0 & p_3 \\ 1 & q^1 & -p_3 & 0 \end{bmatrix}, \quad \theta_{1,R} = -\frac{1}{2} \begin{bmatrix} 0_2 & I_2 \\ -I_2 & 0_2 \end{bmatrix}$$

so that $\theta_{1,R}$ attains in the variables (q^1, q^2, p_2, p_3) the canonical form. An easy calculation yields that the Hamiltonians H_i restricted to \mathcal{S} become

$$\begin{aligned} H_{1,R}(q^1, q^2, p_2, p_3) &= H_1|_{\mathcal{S}} = -q^1 p_2 p_3 + \frac{1}{2} p_2^2 - q^2 p_3^2 - 20(q^1)^3 q^2 \\ &\quad + 12q^1 (q^2)^2 + 3(q^1)^5 \\ H_{2,R}(q^1, q^2, p_2, p_3) &= H_2|_{\mathcal{S}} = (q^2 + \frac{1}{2}(q^1)^2) p_2 p_3 + 9(q^1)^2 (q^2)^2 \\ &\quad + \frac{5}{2}(q^1)^4 q^2 - \frac{5}{4}(q^1)^6 - 2(q^2)^3 \\ H_{3,R}(q^1, q^2, p_2, p_3) &= H_3|_{\mathcal{S}} = 0 \end{aligned}$$

while the functions $\rho_{i,R}$ are: $\rho_{1,R} = -q^1$, $\rho_{2,R} = \frac{1}{4}(q^1)^2 + \frac{1}{2}q^2$, $\rho_{3,R} = 0$. Thus, we have obtained on \mathcal{S} a new separable quasi-bi-Hamiltonian system of the form (20) with $n = 2$. This concludes our first example.

As a second example we will consider a separable (quasi)-bi-Hamiltonian system with a non-trivial function f . More specifically, we shall consider a Lagrangian system of second order Newton equations

$$\begin{aligned} q_{,tt}^1 &= 16q^1(1 + q^2) \\ q_{,tt}^2 &= 8(q^1)^2 + 4(q^3)^2 + 64q^2 + 48(q^2)^2 + 4c + 24 \\ q_{,tt}^3 &= 4q^3(3 + 4q^2) \end{aligned}$$

This system is a part of a quasi-bi-Hamiltonian chain with Hamiltonians

$$\begin{aligned} H_1 &= p_1^2 + p_2^2 + p_3^2 - 4(q^1)^2(1 + q^2) - (q^3)^2(3 + 4q^2) \\ &\quad - 4(3q^2 + 4(q^2)^2 + 2(q^2)^3) \\ H_2 &= -(2q^2 + 1)p_1^2 + 2q^1 p_1 p_2 - p_2^2 + 2q^3 p_2 p_3 - 2p_3^2(q^2 + 1) \\ &\quad - (q^1)^2((q^1)^2 + 2(q^3)^2 + 4(q^2)^2 + 4q^2 + 2) - (q^3)^2((q^3)^2 + 4(q^2)^2 + 2q^2) \\ &\quad + 8(q^2)^3 + 16(q^2)^2 + 12q^2 \\ H_3 &= -(q^3)^2 p_1^2 + 2q^1 q^3 p_1 p_3 - 2q^3 p_2 p_3 + (1 + 2q^2 - (q^1)^2) p_3^2 \\ &\quad + ((q^1)^2 + 4(q^2)^2 + (q^3)^2 + 6q^2 + 3)(q^3)^2, \end{aligned} \tag{43}$$

with the canonical Poisson tensors θ_0 and

$$\theta_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & q^1 & 0 \\ 0 & 0 & 0 & q^1 & 2q^2 + 1 & q^3 \\ 0 & 0 & 0 & 0 & q^3 & 0 \\ -1 & -q^1 & 0 & 0 & -p_1 & 0 \\ -q^1 & -2q^2 - 1 & -q^3 & p_1 & 0 & p_3 \\ 0 & -q^3 & 0 & 0 & -p_3 & 0 \end{bmatrix}$$

and with the functions

$$\rho_1(q) = -2(q^2 + 1) \quad , \quad \rho_2(q) = 1 - (q^3)^2 - (q^1)^2 + 2q^2 \quad , \quad \rho_3(q) = (q^3)^2$$

The above system is a particular example of so called bi-cofactor systems [27], [28], [23], [29], [30] separated recently in [15]. In Darboux-Nijenhuis coordinates (λ, μ) it turns out that now $f(\lambda^i) = 4\lambda^i(\lambda^i - 1)$ and $\gamma_i(\lambda^i) = (\lambda^i)^5$. This means that in order to find the constraint $\varphi_2''(q, p)$ we have to use the formula (36) with $\varphi_1''(q) = (q^3)^2$. Plugging these functions into (36) and using the above forms of G and F we get a rather complicated expression for φ_2''

$$\varphi_2''(q, p) \equiv v_3(q, p) = \frac{1}{4} \left(p_1 \frac{(q^1)^2 + (q^3)^2}{q^1} - p_2 + p_3 \left(-q^3 + \frac{2q^2 + (q^1)^2 - 1}{q^3} \right) \right)$$

Notice that the constraint φ_2'' seems to have a singularity on the surface $\varphi_1'' = 0$. However, it turns out that on the submanifold $\mathcal{S} = \{\varphi_1 = \varphi_2 = 0\}$ we have $p_3 = 0$ and the singularity disappears (see below). Having obtained the constraints φ_1'' and φ_2'' in (q, p) variables we now proceed as in the first example. First we find

$$\theta_{0,D} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & \zeta_1 \\ 0 & 0 & 0 & 0 & 1 & \zeta_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & \zeta_3 \\ 0 & -1 & 0 & 0 & 0 & \zeta_4 \\ -\zeta_1 & -\zeta_2 & 0 & -\zeta_3 & -\zeta_4 & 0 \end{bmatrix}$$

(where ζ_i are some rational functions of q and p that do not vanish on \mathcal{S}) and

$$\theta_{1,D} = \begin{bmatrix} 0 & 0 & 0 & 1 & \frac{(q^3)^2}{q^1} + q^1 & 0 \\ 0 & 0 & 0 & q^1 & 2q^2 + 1 & q^3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -q^1 & 0 & 0 & p_1 \left(\frac{(q^3)^2}{(q^1)^2} - 1 \right) & 0 \\ -\frac{(q^3)^2}{q^1} - q^1 & -2q^2 - 1 & 0 & p_1 \left(1 - \frac{(q^3)^2}{(q^1)^2} \right) & 0 & 2p_3 \\ 0 & -q^3 & 0 & 0 & -2p_3 & 0 \end{bmatrix}$$

(notice that in both cases q^3 is a Casimir function). It can be checked that the Schouten bracket of $\theta_{0,D}$ and $\theta_{1,D}$ is equal to zero: $[\theta_{0,D}, \theta_{1,D}]_{\mathcal{S}} = 0$ so that

the operators $\theta_{0,D}$ and $\theta_{1,D}$ are compatible. Before projecting $\theta_{0,D}$ and $\theta_{1,D}$ onto \mathcal{S} we perform a transformation of variables to Casimir variables

$$(q^1, q^2, \varphi_1''(q), p_1, p_2, \varphi_2''(q, p)) \quad (44)$$

since this time it is easier to eliminate q^3 and p_3 from the system of equations $\varphi_1'' = \varphi_1''(q)$, $\varphi_2'' = \varphi_2''(q, p)$. Explicitly we have

$$\begin{aligned} q^3 &= \pm \sqrt{\varphi_1''} \\ p_3 &= \sqrt{\varphi_1''} \frac{4\varphi_2'' q^1 + \varphi_1'' p_1 + (q^1)^2 p_1 - q^1 p_2}{2q^1 q^2 + q^1 - q^1 \varphi_1'' - (q^1)^3} \end{aligned} \quad (45)$$

As we have mentioned, we have now $p_3|_{\mathcal{S}} = 0$. We will thus parametrize our submanifold \mathcal{S} by the coordinates (q^1, q^2, p_1, p_2) . After expressing the operators $\theta_{i,D}$ in the variables (44) with the help of relations (45) we can easily project these operators on \mathcal{S} . As the result we get a new (reduced) quasi-bi-Hamiltonian chain of the form (20) with $n = 2$, which is the variables (q^1, q^2, p_1, p_2) is determined by

$$\theta_{0,R} = \begin{bmatrix} 0_2 & I_2 \\ I_2 & 0_2 \end{bmatrix}, \quad \theta_{1,R} = \begin{bmatrix} 0 & 0 & 1 & q^1 \\ 0 & 0 & q^1 & 2q^2 + 1 \\ -1 & -q^1 & 0 & -p_1 \\ -q^1 & -2q^2 - 1 & p_1 & 0 \end{bmatrix}$$

and by the restricted Hamiltonians H_i

$$\begin{aligned} H_{1,R} &= H_1|_{\mathcal{S}} = p_1^2 + p_2^2 - 4(q^1)^2(1 + q^2) - 4(3q^2 + 4(q^2)^2 + 2(q^2)^3) \\ H_{2,R} &= H_2|_{\mathcal{S}} = -(1 + 2q^2)p_1^2 + 2q^1 p_1 p_2 q - p_2^2 \\ &\quad - (q^1)^2((q^1)^2 + 2(q^3)^2 + 4(q^2)^2 + 4q^2 + 2) \\ &\quad + 8(q^2)^3 + 16(q^2)^2 + 12q^2 \\ H_{3,R} &= H_3|_{\mathcal{S}} = 0 \end{aligned}$$

while the functions $\rho_{i,R}$ are given by $\rho_{1,R} = -2c(q^2 + 1)$, $\rho_{2,R} = 1 + 2q^2 - (q^1)^2$, $\rho_{3,R} = 0$.

Notice that when we take $\varphi_2''(q, p) = 2q^3 p_3$, i.e. the differential consequence of $\varphi_1''(q, p) = (q^3)^2$, then $\{\varphi_1'', \varphi_2''\}_{\theta_1} = 0$ and the Dirac deformation $\theta_{1,D}$ does not exist (see Remark 10).

7 Conclusions

In this article we have focused on the problem of Dirac deformation and Dirac reduction of Poisson operators and Poisson pencils. We have presented a procedure of performing Dirac reduction in quasi-bi-Hamiltonian systems of Benenti type that do not destroy the separability of these systems and that moreover do not change the separation variables. The method presented is not, however,

general in the sense that it does not provide us with all such reductions, but only with their subclass. Moreover, for the moment the procedure works only for those systems, for which the functions f_i are all equal, i.e. when f_i does not depend on i . Last but not least, the presented procedure is valid only in case when one of the Poisson structures of the system is canonical - the non-canonical case must be studied separately. These issues will be addressed in a separate paper.

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